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Sufficiency in Linear Time Optimal Control*

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INTRODUCTION

Let z be a once continuously differentiable mapping from $[0, \infty)$ into E^n (Euclidean n dimensional space) and

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \qquad x(0) = x^0, \qquad \left(\dot{x}(t) = \frac{dx(t)}{dt}\right) \tag{1}$$

represent a linear control system, where, mainly for notational convenience, it is assumed that the components u_i, $i = 1, \cdots, r$, of the control vector u may be chosen from the class of Lebesgue measurable functions satisfying $| u_i(t) | \leqslant 1$, $t \geqslant 0$. A and B are, respectively, $n \times n$ and $n \times r$ matrix valued functions, summable on compact real intervals. The problem considered will be to determine sufficient conditions that a control u^* "steer" the corresponding solution $x(\cdot; u^*)$ of (1) to the target z in minimum time $t \geqslant 0$.

Following closely the notation of LaSalle [1], let Ω denote the class of all measurable, r vector valued functions u with $| u_j(t) | \leqslant 1, t \geqslant 0, j = 1, \cdots, r$; $X(t)$ a fundamental solution of $\dot{x}(t) = A(t)\,x(t)$; and

$$\mathscr{A}(t) = \{x(t; u) \in E^n : u \in \Omega\}$$

where,

$$x(t; u) = X(t)\,x^0 + X(t) \int_0^t X^{-1}(\tau)\,B(\tau)\,u(\tau)\,d\tau. \tag{2}$$

The set $\mathscr{A}(t)$, termed the attainable set at t, is known to be convex, compact; and the set valued function $\mathscr{A}$ is continuous when considered as a mapping of the positive real line into the space of nonempty, compact subsets of E^n endowed with the Hausdorff metric topology.

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The results of [1], which immediately extend to the case when A and B are summable rather than continuous, show that if there exists a t_1 such that $z(t_1) \in \mathscr{A}(t_1)$, then letting $t^* = \inf. \{t : z(t) \in \mathscr{A}(t)\}$, there exists a $u^* \in \Omega$ such that $x(t^*; u^*) = z(t^*)$. Furthermore $z(t^*) \in \partial\mathscr{A}(t^*)$ ($\partial\mathscr{A}(t^*)$ denotes the boundary of $\mathscr{A}(t^*)$) and u^* satisfies the necessary condition

$$\langle \xi, x(t^*; u) - x(t^*; u^*) \rangle \leqslant 0, \qquad u \in \Omega, \tag{3}$$

where the brackets denote inner product and ξ is a normal to a support plane of $\mathscr{A}(t^*)$ at $z(t^*)$ directed away from $\mathscr{A}(t^*)$. Condition (3) insures that the trajectory corresponding to the optimal control u^* lies on the boundary of the attainable set, and has the equivalent formulation that u^* have the form $u^*(t) = \operatorname{sgn}[\xi X(t^*) X^{-1}(t) B(t)]$ whenever the right side is defined.

The time t^* will be defined to be a *local minimum time* and u^* a *local optimal control* if $x(t^*; u^*) = z(t^*)$, while for some $\delta > 0$, $z(t) \notin \mathscr{A}(t)$ for $t^* - \delta \leqslant t < t^*$. On the other hand, t^* will be called *the minimum time* and u^* *the optimal control* if $x(t^*; u^*) = z(t^*)$ and $z(t) \notin \mathscr{A}(t)$ for $0 \leqslant t < t^*$.

For optimal control problems having fixed terminal time and satisfying certain convexity properties, sufficient conditions are given in [2].

1. A General Sufficient Condition

A further necessary condition, resembling the transversality condition [3, pages 58-62 and 108-114] will first be derived. It should be noted that in [3], the derivation of transversality tacitly assumes that the right sides of the differential equations possess a time derivative, which is not the present case. It should be remarked, however, that this assumption in [3] is seemingly only made for convenience of presentation in the nonlinear problem. Another difference is that while the classical transversality condition tells how an optimal trajectory strikes a terminal manifold, the following result can be viewed geometrically as a condition on how the terminal manifold z strikes the attainable set; i.e., if t^* is an optimal time, then as t increases to t^* the target $z(t)$ must not approach the boundary of the attainable set from within this set.

The time optimal problem stated in the introduction has the following convenient equivalent formulation. Let

$$w(t) = X^{-1}(t)\, z(t) - x^0, \qquad y(t; u) = \int_0^t X^{-1}(\tau)\, B(\tau)\, u(\tau)\, d\tau,$$

and

$$R(t) = \{y(t; u) : u \in \Omega\}.$$

Then

$$w(t) \in R(t) \Leftrightarrow z(t) \in \mathscr{A}(t).$$

$R(t)$ is convex, compact and when considered as a set valued function of t, is continuous in the Hausdorff metric topology. Also

$$z(t^*) \in \partial\mathscr{A}(t^*) \Leftrightarrow w(t^*) \in \partial R(t^*)$$

and corresponding to each outward normal ξ to $\partial\mathscr{A}(t^*)$ at $z(t^*)$ will be a vector $\eta = \xi X(t^*)$ which is a normal to a support plane of $R(t^*)$ at $w(t^*)$ and directed to a side of the support plane containing no points of $R(t^*)$. Let N denote the set of unit vectors η having this property.

Condition (3) has the equivalent formulation that if u^* is a local optimal control, it is necessary that there exist a nonzero vector η such that $\langle \eta, y(t^*; u) - y(t^*; u^*) \rangle \leqslant 0$, $u \in \Omega$. (If $y(t^*; u^*) = w(t^*)$, $\eta \in N$.) From this it follows that a necessary condition for u^* to be a local optimal control and t^* a local optimal time is that there exist a nonzero vector η such that

$$\langle \eta, y(t; u) - y(t; u^*) \rangle \leqslant 0 \qquad \text{for each} \qquad t \in [0, t^*], \qquad u \in \Omega. \tag{4}$$

The transversality condition given in the following theorem will be shown to have the geometric interpretation that with increasing t, $w(t)$ is not approaching the boundary of $R(t^*)$ from within $R(t)$. The same interpretation therefore is valid with $w(t)$, $R(t)$ replaced by $z(t)$, $\mathscr{A}(t)$, respectively.

THEOREM 1. *Let $u^* \in \Omega$ be such that $x(t^*; u^*) = z(t^*)$ for some $t^* > 0$, and in the Lebesgue set of the measurable functions A, B and u^*. A necessary condition for u^* to be a local optimal control and t^* a local optimal time is that there exist a nonzero vector ξ satisfying (3), and furthermore*

$$\xi A(t^*)\, x(t^*; u^*) + \xi B(t^*)\, u^*(t^*) - \xi \cdot \dot{z}(t^*) \geqslant 0. \tag{5}$$

(For notational convenience $\xi \cdot \eta$ and $\langle \xi, \eta \rangle$ will both be used to denote inner product.)

PROOF. Assume u^* is a local optimal control and

$$x(t^*; u^*) = z(t^*) \in \partial\mathscr{A}(t^*),$$

therefore $y(t^*; u^*) = w(t^*) \in \partial R(t^*)$. The necessary condition (3) was proved in [1], and it was shown that ξ could be interpreted as a normal to any support plane to $\mathscr{A}(t^*)$ at $z(t^*)$, directed away from $\mathscr{A}(t^*)$. We will proceed by first giving a geometric motivation for condition (5).

Corresponding to each $\eta \in N$, there will be a support plane $H(\eta)$ to the convex set $R(t^*)$ at $w(t^*)$. Let $h^1(\eta), \cdots, h^{n-1}(\eta)$ denote $n - 1$ (remember $R(t^*) \subset E^n$) linearly independent unit vectors each orthogonal to η, which then determine the support plane $H(\eta)$. The vectors

$$\{(1, \dot{y}(t^*; u^*)), (0, h^1(\eta)), \cdots, (0, h^{n-1}(\eta))\}$$

are linearly independent and determine a hyperplane $P(\eta)$ at the point $(t^*, w(t^*))$ in the $n + 1$ dimensional (t, x) space. A normal to $P(\eta)$ directed (at least locally) away from the attainable cone $\bigcup_{t \geqslant 0} R(t)$, at $(t^*, w(t^*))$, is given by $(-\eta \cdot \dot{y}(t^*; u^*), \eta)$. The condition (5) is then equivalent to the statement: there exists at least one $\eta \in N$ such that

$$\langle (1, \dot{w}(t^*)), (-\eta \cdot \dot{y}(t^*; u^*), \eta) \rangle = \langle \eta, \dot{w}(t^*) - \dot{y}(t^*; u^*) \rangle \leqslant 0, \qquad (6)$$

as is easily verified by direct substitution, i.e. with increasing t, $w(t)$ is not approaching the boundary of $R(t^*)$ from within $R(t)$.

At this point the actual proof begins. The assumption

$$\langle \eta, \dot{w}(t^*) - \dot{y}(t^*; u^*) \rangle > 0 \qquad \text{for all} \qquad \eta \in N$$

will now be made and it will be shown that this leads to a contradiction to t^* being a local minimum time and u^* a local optimal control.

N may be identified with a subset of the $n - 1$ sphere. Since $R(t^*)$ is compact and convex, N is compact and the minimum of $\langle \eta, \dot{w}(t^*) - \dot{y}(t^*; u^*) \rangle$ taken over $\eta \in N$ is positive. Let

$$m = \min_{\eta \in N} \langle \eta, \dot{w}(t^*) - \dot{y}(t^*; u^*) \rangle.$$

Since t^* is in the Lebesgue set of A, B and u^*, there is a $\delta > 0$ such that

$$\begin{aligned} \langle \eta, y(t; u^*) - w(t) \rangle &= \int_t^{t^*} \langle \eta, \dot{w}(\tau) - \dot{y}(\tau; u^*) \rangle \, d\tau \\ &\geqslant [(t^* - t)\, m + o(t^* - t)] > 0 \qquad (7) \end{aligned}$$

for $t \in [t^* - \delta, t^*)$, $\eta \in N$.

Since u^* is assumed to be an optimal control, the necessary condition (4) must be satisfied which implies $y(t; u^*) \in \partial R(t)$, and the same set N constitutes the set of outward normals to hyperplanes of support of $R(t)$ at $y(t; u^*)$. Inequality (7) now implies that the interior of $R(t)$ (int. $R(t)$) is not empty. Indeed, suppose $R(t)$ were contained in a subspace of dimension less than n and would therefore have an empty n dimensional interior. There would then be an $\eta' \in N$ such that the corresponding support plane contains $R(t)$. Then $-\eta'$ also belongs to N but we cannot have both $\langle \eta', y(t; u^*) - w(t) \rangle$ and $\langle -\eta', y(t; u^*) - w(t) \rangle$ positive. Since int. $R(t)$ is nonempty and $R(t) \subset R(t^*)$ for $t \leqslant t^*$, the int. $R(t^*)$ is not empty.

Inequality (7) also shows that for $t \in [t^* - \delta, t^*)$, $w(t)$ lies on the same side of every hyperplane of support to $R(t)$ at $y(t; u^*)$ as does $R(t)$. From (7), (4) and noting that $y(t; u^*) \in R(t^*)$ for $t \leqslant t^*$,

$$
\begin{aligned}
0 < [(t^* - t)\, m + o(t^* - t)] &\leqslant \langle \eta, y(t; u^*) - y(t^*; u^*) + y(t^*; u^*) - w(t) \rangle \\
&= \langle \eta, y(t^*; u^*) - w(t) \rangle + \langle \eta, y(t; u^*) - y(t^*; u^*) \rangle \\
&\leqslant \langle \eta, y(t^*; u^*) - w(t) \rangle \qquad \text{for} \qquad t \in [t^* - \delta, t^*), \qquad \eta \in N. \qquad (8)
\end{aligned}
$$

This shows that $w(t)$ lies on the same side of every hyperplane of support to $R(t^*)$ at $y(t^*; u^*)$ as does $R(t^*)$, for $t \in [t^* - \delta, t)$. This does not yet imply $w(t') \in R(t')$ for some $t' \in [t^* - \delta, t^*)$, which is the contradiction we seek.

If it can be shown that there exists a $\delta' > 0$ such that $w(t) \in \text{int.}\, R(t^*)$ for $t \in [t^* - \delta', t^*)$ the proof would be complete, since the continuity of the set function R would then imply there exists an ϵ_1 satisfying $0 < \epsilon_1 \leqslant \delta'$ such that $w(t^* - \epsilon_1) \in R(t^* - \epsilon_1)$, contradicting the local optimality of u^*. This will now be shown to complete the proof.

Since t^* is a Lebesque point of A, B, and u^*, there exists a $\delta_1 > 0$ such that for $t \in [t^* - \delta_1, t^*)$

$$
\begin{aligned}
&| y(t^*; u^*) - w(t) | - | \dot{w}(t^*) | \, | t^* - t | \\
&\leqslant | y(t^*; u^*) - w(t^*) - \dot{w}(t^*)(t^* - t) | = \left| \int_t^{t^*} (\dot{w}(\tau) - \dot{w}(t^*))\, d\tau \right| \\
&\leqslant \int_t^{t^*} | \dot{w}(\tau) - \dot{w}(t^*) | \, d\tau = o(t^* - t)
\end{aligned}
$$

or

$$
| y(t^*; u^*) - w(t) | \leqslant | \dot{w}(t^*) | \, | t^* - t | + o(t^* - t), \qquad t \in [t^* - \delta_1, t^*). \tag{9}
$$

Letting $\delta_2 = \min \{\delta_1, \delta\}$, inequality (8) can be written as

$$
| y(t^*; u^*) - w(t) | \cos(\eta, y(t^*; u^*) - w(t)) \geqslant [(t^* - t)\, m + o(t^* - t)] > 0
$$

for

$$
t \in [t^* - \delta_2, t^*), \qquad \eta \in N. \tag{10}
$$

Since the original assumption $\langle \eta, \dot{w}(t^*) - \dot{y}(t^*; u^*) \rangle > 0$ for all $\eta \in N$ implies $| \dot{w}(t^*) | \neq 0$, (9) and (10) can be combined to give

$$
\cos(\eta, y(t^*; u^*) - w(t)) \geqslant \frac{m}{2 \, | \dot{w}(t^*) |} > 0 \tag{11}
$$

for all $\eta \in N$ and $t \in [t^* - \delta_3, t^*]$, where $0 < \delta_3 \leqslant \delta_2$.

Geometrically this can be interpreted as follows. Consider a unit $n - 1$ sphere S^{n-1} with center at $y(t^*; u^*)$. The set N is a compact convex subset of S^{n-1}. For each $\eta \in N$ the set

$$
C(\eta) = \left\{ w \in E^n : \cos(\eta, y(t^*; u^*) - w) \geqslant \frac{m}{2 \, | \dot{w}(t^*) |} > 0 \right\}
$$

is a cone with vertex at $y(t^*; u^*)$ which intersects S^{n-1} in a spherical cap centered about the point $-\eta \in S^{n-1}$, while the property $m/(2 \mid \dot{w}(t^*) \mid) > 0$ insures that the spherical cap is properly contained in a hemisphere. Let $C = \bigcap_{\eta \in N} C(\eta)$. C is a nonempty $(y(t^*; u^*) \in C)$ cone and (11) implies $w(t) \in C$ for each $t \in [t^* - \delta_3, t^*)$. Also, a sufficiently small segment of the tip of the cone C, with the exception of the vertex $y(t^*; u^*)$, belongs to the interior of $R(t^*)$. Indeed, C being the intersection of convex sets is itself a convex set, and the set $\Gamma \subset S^{n-1}$ of unit outward normals to hyperplanes of support to C at $y(t^*; u^*)$ properly contains a neighborhood of N. But (8) implies $w(t) \neq y(t^*; u^*)$ for $t \in [t^* - \delta_3, t^*)$, therefore for these values of t, $w(t)$ is in the interior of $R(t^*)$, which completes the proof.

COROLLARY 1.1. *(A sufficient Condition). Let u^* be an admissible control such that $x(t^*; u^*) = z(t^*)$. A sufficient condition that u^* be a local optimal control and t^* a local minimum time is that t^* be in the Lebesque set of A, B, and u^*, and u^* satisfy the necessary condition (3) and the necessary condition (5) with the latter having inequality replaced by strict inequality.*

PROOF. Since t^* is in the Lebesgue set of A, B, and u^*, condition (5) with strict inequality implies there exists a $\delta > 0$ and $\eta' \in N$ such that for $t \in [t^* - \delta, t^*)$, $\int_t^{t^*} \langle \eta', \dot{w}(\tau) - \dot{y}(\tau; u^*)\rangle \, d\tau < 0$ or

$$\langle \eta', y(t; u^*) - w(t)\rangle < 0, \qquad t \in [t^* - \delta, t^*). \tag{12}$$

Now suppose t^* is not a local minimum time and u^* not a local optimal control. Then there must be $t \in [t^* - \delta, t^*)$ such that $w(t) \in R(t)$ hence a control u such that $\langle \eta', y(t; u) - w(t)\rangle = 0$. But this contradicts at least one of the inequalities (12) and (4), for combining them gives $\langle \eta', y(t; u) - w(t)\rangle < 0$ for all $u \in \Omega$, $t \in [t^* - \delta, t^*)$.

2. SUFFICIENT CONDITIONS WHEN THE TARGET IS A FIXED POINT

In [1, Theorem 5], LaSalle showed that a sufficient condition that a control u^* of the form $u^*(t) = \operatorname{sgn} [\xi X(t^*) X^{-1}(t) B(t)]$ for some unit vector ξ, be an optimal control and t^* the minimum time when the target is the origin, is that $x(t^*; u^*) = 0$ and $\xi X(t^*) X^{-1}(t) B(t) \neq 0$, the zero vector, except on a set of measure zero.

It should be noted that this result depends strongly on the target being the origin. For example, consider the one dimensional problem of hitting the fixed point $z = \frac{1}{4}$ in minimum time by a trajectory of

$$\dot{x}(t) = -x(t) + e^{-t}u(t), \qquad x(0) = 0, \qquad \text{and} \qquad | u(t) | \leq 1.$$

The optimal control is $u^*(t) \equiv 1$, while the corresponding solution is

$x(t; u^*) = te^{-t}$ which assumes the value $\frac{1}{4}$ for two distinct values of t, say t_1 and t_2, $t_1 < t_2$. The control is of the form

$$u^*(t) = \operatorname{sgn}[\exp(-t_i)] = \operatorname{sgn}[\xi X(t_i) X^{-1}(t) B(t)] \qquad \text{for} \qquad i = 1, 2,$$

but t_2 is a maximum, not a minimum time.

To alleviate the difficulty pointed out by this example, define: *$\mathscr{A}$ is expanding at t^* if there exists a $\delta > 0$ such that for every $t \in [t^* - \delta, t^*)$ there is an $\epsilon = \epsilon(t) > 0$ such that $\mathscr{A}(t^*)$ contains the $\epsilon(t)$ neighborhood of $\mathscr{A}(t)$.* If $\mathscr{A}$ is expanding for arbitrary $t^* > 0$, we shall say $\mathscr{A}$ is expanding. It is evident that if the target z is stationary and $\mathscr{A}$ is expanding, then z will belong to the boundary of $\mathscr{A}(t)$ for at most one value of t. Since condition (3) assures $x(t^*; u^*)$ belongs to the boundary of $\mathscr{A}(t^*)$, the following lemmas are immediate.

LEMMA 1. *A sufficient condition for t^* to be a local minimum time and u^* a local optimal control for stationary target z is that $x(t^*; u^*) = z$, u^* have the form $u^*(t) = \operatorname{sgn}[\xi X(t^*) X^{-1}(t) B(t)]$ for some unit vector ξ, and $\mathscr{A}$ be expanding at t^*.*

LEMMA 2. *A sufficient condition for t^* to be the minimum time and u^* the optimal control, for stationary target z, is that u^* have the form $\operatorname{sgn}[\xi X(t^*) X^{-1}(t) V(t)]$ and $\mathscr{A}$ be expanding.*

We will next determine conditions which insure that $\mathscr{A}$ is expanding under the simplifying assumption that $x^0 = 0$. For an arbitrary unit vector ξ, define

$$v(\tau, t, \xi) = \xi X(t) X^{-1}(\tau) B(\tau). \tag{13}$$

This implies one should choose $u^*(\tau) = \operatorname{sgn} v(\tau, t, \xi)$ for $0 \leqslant \tau \leqslant t$ to arrive at a point $x(t; u^*) \in \mathscr{A}(t)$ which has maximum projection on ξ.

Let

$$| v(\tau, t, \xi) | = \sum_{j=1}^{r} | v_j(\tau, t, \xi) | .$$

LEMMA 3. *A sufficient condition that $\mathscr{A}$ be expanding at t^*, for initial data $x^0 = 0$, is that there exist a $\delta > 0$ such that*

$$\int_0^t | v(\tau, t^*, \xi) | \, d\tau < \int_0^{t^*} | v(\tau, t^*, \xi) | \, d\tau$$

for all unit vectors ξ and $t \in [t^* - \delta, t^*)$.

PROOF. Let $\mathscr{N}(\mathscr{A}(t), \epsilon)$ denote an ϵ neighborhood of the set $\mathscr{A}(t)$. Now ξ is an arbitrary unit vector, i.e. an element of the $(n - 1)$ sphere which is compact, and for fixed $t \in [t^* - \delta, t^*)$, $\int_0^t |v(\tau, t, \xi)| \, d\tau$ is a continuous function of ξ. Therefore there exists an $\epsilon = \epsilon(t) > 0$ such that

$$\int_0^t |v(\tau, t, \xi)| \, d\tau + \epsilon(t) \leqslant \int_0^{t^*} |v(\tau, t^*, \xi)| \, d\tau \qquad \text{for all} \qquad \xi.$$

But this states exactly that

$$\max_{x \in \mathscr{A}(t)} \langle x, \xi \rangle + \epsilon(t) \leqslant \max_{x \in \mathscr{A}(t^*)} \langle x, \xi \rangle,$$

which assures that $\mathscr{A}(t^*)$ contains $\mathscr{N}(\mathscr{A}(t), \epsilon(t))$ for $t \in [t^* - \delta, t^*)$, i.e. $\mathscr{A}$ is expanding at t^*.

LEMMA 4. *If $\int_0^t |v(\tau, t, \xi)| \, d\tau$ is a strictly increasing function of t for arbitrary ξ, and $x^0 = 0$, then $\mathscr{A}$ is expanding.*

The proof is an immediate consequence of Lemma 3.

THEOREM 2. *A sufficient condition that t^* be the optimal time and u^* the optimal control for the problem of hitting the stationary target z in minimum time by a trajectory of the system $\dot{x}(t) = A(t)\,x(t) + Bu(t)$, $x(0) = 0$, B a constant matrix, is that $x(t^*; u^*) = z$, $u^*(t)$ have the form* sgn. $[\xi X(t^*)\, X^{-1}(t) B]$, *and there exist no unit vector η such that $\eta X(t)\, B = 0$ (the zero vector) on a set of positive measure in* $[0, t^*]$.

PROOF. In order to apply Lemma 4, it will be shown that

$$\frac{d}{dt} \int_0^t |v(\tau, t, \xi)| \, d\tau$$

is positive almost everywhere in t for arbitrary ξ. Indeed, with B constant,

$$\frac{d}{dt} \int_0^t |v(\tau, t, \xi)| \, d\tau = |v(t, t, \xi)| + \int_0^t \frac{d}{dt} |v(\tau, t, \xi)| \, d\tau$$

$$= |v(t, t, \xi)| - \int_0^t \frac{d}{d\tau} |v(\tau, t, \xi)| \, d\tau = |v(0, t, \xi)| \, .$$

But $v(0, t, \xi) = \xi X(t)\, B$, and by assumption this is nonzero for almost all t and arbitrary unit vector ξ. Thus $\mathscr{A}$ is expanding, and the conclusion of the theorem follows from Lemma 2.

It is of interest to note that all of the sufficiency conditions stated require directly, or imply, that the interior of $\mathscr{A}(t^*)$ is nonempty .This is a controllability conclusion. Corollary 1.1 requires interior $\mathscr{A}(t^*)$ nonempty by virtue of strict inequality in (5). The result of LaSalle referred to in the beginning of Section 2 requires interior $\mathscr{A}(t^*)$ nonempty by virtue of the condition $\xi X(t^*)\, X^{-1}(t)\, B(t) \neq 0$, except on a set of measure zero, which is equivalent to complete controllability as shown in [1]. The notion of $\mathscr{A}$ expanding, by definition, requires interior $\mathscr{A}(t^*)$ to be nonempty, and the conditions of Theorem 2 imply $\mathscr{A}$ is expanding.

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